Local Low Dimensionality and Relation to Effects Of Targeted Weather Observations

D. J. Patil*, Istvan Szunyogh*, Aleksey V. Zimin*, Brian R. Hunt*, Edward Ott*, Eugenia Kalnay* and James A. Yorke*

*University of Maryland

Abstract. A statistic, the BV (bred vector) dimension, is introduced to measure the effective local finite-time dimensionality of a spatiotemporally chaotic system. It is shown that the Earth’s atmosphere often has low BV-dimension. The implications for improving weather forecasting through data assimilation and targeted observations are discussed.

INTRODUCTION

A key question in the understanding of local instabilities that lead to the breakdown in forecasts, is the complexity of their dynamics (behavior). In this paper, we demonstrate that, in spite of the atmosphere’s high dimensionality, in a suitable sense the local finite-time atmospheric dynamics is often low dimensional, and we conjecture that these low dimensional regions have a strong relationship to dynamical instabilities.

This paper outlines results that are detailed in [1, 2, 3, 4, 5] as well as several preliminary results which we plan to include in forthcoming papers. Section 2 introduces the method of breeding [6, 7] to produce an ensemble of forecasts from different initial conditions (see [8] and [9] and references within for a review of different methods to generate the initial conditions). Section 3 describes the bred vector ensembles that we utilize. Results regarding the local dimensionality from a similar ensemble (National Weather Service’s National Centers for Environmental Prediction’s Ensemble Forecasting System) have been previously discussed in [1, 2]. The major difference between the two ensembles is that the results presented here are based on an ensemble that uses many (six times) more bred vectors. A main aim of this work is to investigate the robustness of the results of [1, 2] to changes in the ensemble size. In Section 4 we discuss the statistic, which we call the Bred Vector dimension (BV-dimension), which effectively determines the dimension of the subspace spanned by members of the ensemble over a geographically localized region. Using our statistic we investigate the Earth’s atmospheric dynamics for the Northern Hemispheric winter. A large membership ensemble run is used to assess the robustness and saturation of the BV-dimension. In Section 6 we provide preliminary results from a study on the relationship between the BV-dimension and the propagation of the impact of targeted observations from the 2000 Winter Storm Reconnaissance Program. We conclude with a discussion about the potential implications of our finding for weather forecasting and data assimilation.
FIGURE 1. Schematic of the breeding cycle for ensemble forecasts

**BREEDING**

The procedure for breeding can be outlined in the following steps: (a) add a perturbation to the base state (usually the atmospheric analysis) at a given time $t_0$; (b) integrate both the base state and the perturbed state forward in time for a period of $t_1 - t_0$ (for this paper $t_1 - t_0 = 24$ hours); (c) at time, $t_1$, subtract the base state from the perturbed state and rescale the difference, so that it has the same norm (for the NWS ensemble system, rotational kinetic energy at approximately 500hPa is used) as the initial perturbation; (d) this rescaled difference is then used as a new perturbation to the base state and the process (a)-(c) are sequentially repeated. The process is illustrated in Figure 1. The iterated difference between the perturbed and the main solution is called the bred vector (see appendix for a theoretical discussion of the breeding process).

**DATA**

The study we report in this paper is based on ensemble forecasts generated by a replica of the operational Global Ensemble Forecasting System at the National Centers for Environmental Prediction (NCEP). We produce six times the number of ensembles that are currently available at 0000 UTC from NCEP by generating 30 pairs of perturbed forecasts by the breeding process for the period of January 1, 2000 to February 28, 2000. Since the perturbed forecasts are integrations of positive and negative initial perturbations, we average these pairs of perturbed forecasts to yield 30 bred vectors. The bred vectors are generated at a T62 spectral resolution and output on a latitude-longitude spatial grid with $2.5\degree x 2.5\degree$ resolution at 21 levels. The perturbed forecasts are rescaled to a regionally varying mask every 24 hours (see [10], [7], and [11] for operational implementations). The rescaling is meant to initialize the magnitudes of the bred vectors to roughly reflect the uncertainties in the analysis of the atmospheric state.

**BRED VECTOR DIMENSION**

In this section, we outline how the bred vectors can be utilized to obtain a useful measure of the dimensionality of the space in which instabilities in the forecasts are likely to grow the fastest. We begin with considering regions at a fixed pressure level of 5 by 5 grid points by choosing a grid point as the center of the region as well as an array of 24 neighbors so that the array best covers a 1100 km x 1100 km square (at high latitudes
some points are skipped in longitude to keep an approximate uniform distance). Given 
N fields at each point (e.g., temperature, wind speeds, etc.), the values at these 25 points, 
for each bred vector, are ordered to form a 25N dimensional column vector which we 
call a local bred vector.

The issue we want to address is the linear independence of the k local bred vectors. 
That is, we want to determine the effective dimensionality of the subspace spanned by 
the local bred vectors. To do this we use empirical orthogonal functions (EOF) [also 
known as principal component analysis] [12]. The underlying concept is to find the 
lowest dimensional subspace that, in a least squares sense, optimally represents the 
majority of the data.

The k local bred vectors form the columns of a 25N \times k matrix, B. The k \times k covariance 
matrix of B is C = B^T B, where B^T is the transpose of B. Since the covariance matrix is 
nonnegative definite and symmetric, its k eigenvalues \lambda_i are nonnegative (\lambda_i \geq 0), and its 
eigenvectors, after multiplying by B and normalizing, form an orthonormal set of vectors 
v_i which span the column space of B. We order the eigenvalues by \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k. 
The singular values of B are \sigma_i = \sqrt{\lambda_i}. The eigenvalues \lambda_i are a measure of the extent 
to which the k column vectors making up B point in the direction v_i, and each \sigma_i^2 = \lambda_i is 
said to represent the amount of variance in the set of the k unit vectors that is accounted 
for by v_i. By identifying how many of the vectors of v_i represent most of the variance 
of B, we can identify an effective dimension spanned by the k local bred vectors. For 
example, if two out of five singular values are zero, then the subspace spanned by the 
k local bred vectors is three dimensional. However, if some of the singular values are 
nonzero but small, the issue becomes more difficult. One option is to use thresholding (as 
is often done when trying to isolate the dominant modes in EOF), but there is difficulty 
in determining the “best” value of the threshold. Instead we opt to define the following 
statistic on the singular values which we call the Bred Vector dimension (BV-dimension):

$$
\psi(\sigma_1, \sigma_2, \ldots, \sigma_k) = \frac{(\sum_{i=1}^{k} \sigma_i)^2}{\sum_{i=1}^{k} \sigma_i^2}.
$$

As examples of the statistic, we describe several cases with k = 5 vectors of unit length. 
If the k local bred vectors comprising B were all the same, then the singular values would 
be \sqrt{5}, 0, 0, 0, 0. This would yield a statistic of \psi(\sqrt{5}, 0, 0, 0, 0) = 1. If the k local bred 
vectors were equally distributed between two orthogonal unit vectors v_1 and v_2, in the 
sense that each one accounts for half the variance, then the singular values would be 
\sqrt{5/2}, \sqrt{5/2}, 0, 0, 0, and our statistic would yield \psi(\sqrt{5/2}, \sqrt{5/2}, 0, 0) = 2. If the 
local bred vectors again lie in the two dimensional subspace spanned by unit vectors 
v_1 and v_2, but the two are not equally represented, then this could give 2 \geq \psi \geq 1. For 
example, if 4 of 5 unit local bred vectors were pointing in the same direction while the 
other pointed in an orthogonal direction, then the singular values would be 2, 1, 0, 0, 0 
and \psi(2, 1, 0, 0, 0) = 1.8. While the dimension of the space spanned by the local bred 
vectors is 2, our statistic gives an intermediate value reflecting the degree of dominance 
of one direction over the other. In general our statistic returns a real value between 1 
and k. Note that while small perturbations due to noise or numerical error will typically 
cause the dimensionality of the space spanned by the k local bred vectors to be k, the
effectiv e dimension $\psi$ may be substantially lower and is insensitive to small changes in $\sigma_i$ due to noise or numerical error.

**REGIONS OF LOW BV-DIMENSION**

In this paper we use two types of local bred vectors. The first is constructed using the east-west and north-south wind components at 250 hPa, 500 hPa, and 850 hPa. Hence, $N = 3 \cdot 2$ and the local bred vectors are $25 \cdot 6 = 150$ dimensional. The second type of bred vector we use is constructed by an *energy rescaling* in order to transform different physical variables (wind, temperature, and surface pressure) to comparable quantities of the same physical dimension [13]. The dimension of these bred vectors is 1525.

An example of the BV-dimension applied to a set of 30 bred vectors from February 6, 2002 is shown in Figure 2. The shading represents the effective dimension of the subspace spanned by the local bred vectors (BV-dimension). Several large regions of low dimensionality (BV-dimension less than 8) are evident. This indicates that in these regions, the local bred vectors effectively span a space of substantially lower dimension than that of the full space (and the number of bred vectors). The regions of low BV-dimension can be shown to be statistically significant and not due to random fluctuations, by applying the BV-dimension to *surrogates* of the data [14] that consist of bred vectors chosen from different days which are substantially far apart (removing temporal correlations). When this is done we observe no regions with low BV-dimension (visual examples for operational data are provided in [1, 2]). Furthermore, we find the regions of low BV-dimension are independent of the choices of variables used to construct the local bred vectors. For example, the local bred vectors constructed from vorticity at 500 hPa identify similar regions of low BV-dimension to the methods described earlier in this section. The primary difference is the magnitude of the BV-dimension in the respective regions of low dimensionality.

It is empirically found that regions of low BV-dimension are robust to increasing the size of the ensemble. More specifically, as the ensemble size is increased, the minimum BV-dimension begins to saturate, while the maximum dimension does not. Figure 3 illustrates this property by plotting the average, over days, of the minimum and maximum BV-dimension for the northern midlatitudes. For comparison, results are also shown for random vectors consisting of Gaussian white noise. While the maximum BV-dimension rapidly increases as the ensemble size is increased, it remains at a large distance from the hypothesis of noise. This result indicates the robustness of the regions of low BV-dimension to the size of the ensemble.

Consistent with [2], large regions of low BV-dimension develop and persist on the order of less than a week and move eastward at about 10°. An example of the time evolution of several regions of low BV-dimension over the northern Pacific ocean are presented in Figure 4. The plots on the left show results from a forecast initiated on February 9, 2000 and figures on the right display results for the corresponding verifying analyses.
FIGURE 2. An example of the spatial variation of BV-dimension for February 6, 2002. The shades represent the effective subspace spanned by the local bred vectors (BV-dimension).

FIGURE 3. The time average of the minimum and maximum BV-dimensions for 30N to 65N as a function of the number of members in the ensemble. The effective saturation of the minimum BV-dimension suggests robustness of regions of low BV-dimension. The vertical bars represent one standard deviation. For comparison, saturation plots for Gaussian noise and surrogates are shown.

RELATIONSHIP BETWEEN THE EFFECTS OF TARGETED WEATHER OBSERVATIONS AND BV-DIMENSION

The National Centers for Environmental Prediction/National Weather Service (NCEP/NWS) implemented an operational targeted weather observation program
FIGURE 4. An example of the development and propagation of several regions of low BV-dimension from a forecast initiated on February 9, 2002 and the corresponding verification times. Contours of BV-dimension 9 to 5 are used to identify the regions labeled A, B, and C.

with the aim of reducing the risk of forecast failures in the prediction of severe winter storms with potentially large impact on society [15]. At NCEP the Ensemble Transform Kalman-Filter (ETKF, [16, 17]) is used to determine the future observational time, \( t_o \), and observational region from where extra observations taken at \( t_o \) are most likely to considerably reduce the error in the prediction of the selected storm at a future verification time, \( t_v (t_v > t_o) \).

While the forecast verification results [18, 19, 11, 15] have provided convincing evidence that the ETKF technique, and its predecessor the Ensemble Transform (ET) tech-
nique [20], are reliable practical tools to target observations, the underlying assumptions of these algorithms have not been rigorously tested. These assumptions are that (i) the analysis error covariance matrix can be well approximated by linearly combining a small ensemble of short-term forecasts valid at the future observational time, $t_o$, and (ii) the forecast error covariance matrix for the grid points within the verification region can be reasonably approximated by applying the same weights to the ensemble of forecasts valid at $t_v$. Given these assumptions, the ETKF technique provides an estimate of the forecast error variance reduction (formally, the reduction in the trace of the forecast error covariance matrix) due to the hypothetical extra observations. The main goal of this paper is to explore whether a small ensemble can represent the uncertainty in the dynamical evolution of the atmospheric flow both at observational and verification times. In other words, we ask: Does the atmosphere show finite-time local low-dimensional chaotic behavior in the regions, where the influence of the targeted data propagates?

To answer the question above, the BV-dimensional is computed based on the two components of the wind vector at the 250, 500 and 850 hPa pressure levels. This choice was made because the same variables are used in the operationally implemented version of the ETKF technique. In what follows, the relationship between the effects of targeted observations and local low dimensionality in the atmosphere is shown for a typical example from the 2000 Winter Storm Reconnaissance program [21].

The BV-dimensional shown in Figure 5 was computed based on the initial perturbations of the experimental ensemble. In this figure the small crosses mark the dropsonde locations (panel a) and the circles mark the verification regions (panels c and e) for a targeting case from the 2000 Winter Storm Reconnaissance program. The figure shows that the targeted observations were taken in an isolated region of local low dimensionality. The dimension of this region, about 5-6, is extremely low considering that the largest possible value of the BV-dimensional for the given ensemble is 30. To track the propagation of the targeted low dimensional spot, an additional forecast ensemble was created. This ensemble was identical to the 30-pair research ensemble, except for that the initial perturbations were set to zero outside of the targeted low dimensional region. The region of local low-dimensionality propagates eastward with the corresponding wave. The propagation of the selected low-dimensional spot is shown in Figure 6. The local dimensions shown in this figure were computed by first adding a small magnitude random noise component to the ensemble perturbations. This was needed to obtain large dimension values in the regions where the magnitude of the ensemble members was smaller than the noise level due to the use of a spectral model. Although this approach slightly increased the local dimension, it has made the tracking of the selected low dimensionality straightforward. After 24 hours the targeted low dimensional feature propagated to the west coast verification region. This is a key finding of our study. This implies that the targeted data had a great potential to reduce the forecast error in the verification region: both the analysis and the forecast uncertainties were confined in low dimensional subspaces, and these low dimensional subspaces were dynamically connected. The contour lines in Figure 1 confirm that along the regions, where the targeted low-dimensionality propagates, the forecast was mainly improved. We have recently developed an algorithm to track the propagation of the error reduction in forecasts due to increased observations in regions of low BV-dimension and we plan to provide details in a future paper.
Upper Tropospheric Wave Packets

The propagation of the local low dimensionality cannot explain how the impact of the dropsonde data propagated to the east coast verification region at 48hr forecast lead time. Szunyogh et. al [19, 21], argued that the impact of the targeted data was propagated from the Pacific to the Atlantic region by packets of short upper tropospheric Rossby waves. In Figure 7 the wave packet envelope function, $A(x)$, is shown, where $A(x)$ is computed using our algorithm described in [4] which assumes that the meridional wind component, $v(x)$ can be expressed as

$$v(x) = A(x) \cos(\phi(x)),$$

where $\phi(x)$ is the spatially varying phase.

Figure 7 shows that an atmospheric wave packet has developed about 12 hours after observational time. The east coast verification region was located at the leading edge of the eastward propagating wave packet at verification time, $t_v$. In order to show that the dropsonde signal propagated as an atmospheric wave packet, the signal, $s(x)$, was also demodulated by substituting $s(x)$ for $v(x)$ in Equation 2. The signal, $s(x)$, is formally defined as the difference between a forecast that was initiated by assimilating all targeted and standard observations and a forecast that was initiated by assimilating only the standard observations. The results demonstrate that after an initial transient, which is not longer than 12 hours, the leading edge of the targeted data impact propagated with the atmospheric wave packet.

A comprehensive analysis of all targeting cases from the WSR00 field program is in progress.

CONCLUSIONS

In this paper our main result is a means of identifying local low dimensional behavior (the BV-dimension) of the atmospheric system. We have provided evidence that these low dimensional, dynamical instabilities (regions of low BV-dimension) cover a significant portion of the globe, that they are intrinsic to the dynamics of the atmospheric system, and that they typically last for several days.

At any given time $t_0$, there is inevitably a discrepancy $\Delta(t_0)$ between the true atmospheric state and its representation in the computer model (analysis). Now consider a later time $t_1 > t_0$, and suppose that in a region of interest there is a low BV-dimension at time $t_1$. This implies that any local discrepancy $\Delta(t_1)$ between the true state and its representation in the computer model (forecast error) lies predominantly in the “unstable subspace”, the space spanned by the few vectors that contribute most strongly to the the low BV-dimension. We conjecture that in many cases this information can yield a substantial improvement in forecasting. In particular, the implication is that the data assimilation algorithm should correct the computer model state by moving it closer to the observations along the direction of the unstable subspace since that is where the true state most likely lies [22]. Current data assimilation techniques (e.g., that used by the NWS) do not take this into account. Based on these results we have developed a data
assimilation algorithm that exploits the local low dimensionality [23, 24]. We plan to report on these results in a later paper.

**APPENDIX: THEORETICAL DISCUSSION OF THE BREEDING PROCESS**

For a $m$-dimensional dynamical system

$$\frac{dx}{dt} = F(x)$$

important dynamical characteristics can be studied by investigating the infinitesimal displacement between two trajectories, $x(t)$ and $x(t) + \delta x(t)$, as well as the behavior of the associated tangent vector $\delta x(t)/|\delta x(0)|$ that evolves according to the equation

$$\frac{d\delta x}{dt} = DF(x(t))\delta x.$$  

Where $DF$ denotes the Jacobian of $F$. The exponential separation of the two trajectories, $x(t)$ and $x(t) + \delta x(t)$, is characterized by the Lyapunov exponent:

$$\Lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\delta x(t)|}{|\delta x(0)|}.$$  

For a given trajectory there are in general $m$ different Lyapunov exponents, $\Lambda_i$, corresponding to different choices of the initial perturbation $\delta x(0)$. The perturbations, $\delta x(t)$, that correspond to a particular Lyapunov exponent, $\Lambda_i$, become concentrated for large $t$ in the direction of a single vector $u_i(t)$, which we call the Lyapunov vector for $\Lambda_i$.

From the practical point of view, the utility of Lyapunov exponents and Lyapunov vectors stems from the fact that, in many nonlinear systems, sufficiently small finite perturbations, $\Delta x(t)$, behave like infinitesimal perturbations, $\delta x(t)$, for finite time, and, over this time interval, these finite perturbations are still large enough to have significant consequences. This situation becomes problematic if in a system like the Earth’s atmosphere many spatial scales are present. As an example, consider the spread of a passive scalar in a turbulent fluid with a Kolmogorov energy wavenumber spectrum, $E(k) \sim k^{-5/3}$, where $k$ denotes the magnitude of the wave number. (The presence of many spatial scales in this case is due to the slow algebraic decay of $E(k)$ with increasing $k$.) For this case the location, $x(t)$, of a passive particle is given by $dx(t)/dt = v(x,t)$, where $v(x,t)$ is the fluid velocity. As shown long ago by [25], on average, $|\Delta x| \sim t^{3/2}$. This is at odds with the expectation of exponential separation obtained by formally following the Lyapunov analysis and writing $d\delta x/dt = \delta x \cdot \nabla v$. The problem is that if $E(k) \sim k^{-5/3}$ applies for $k \to \infty$ (i.e., for arbitrarily small scales), then $\nabla v$ does not exist (i.e., the velocity field is not differentiable). This follows from Parseval’s theorem which implies $\langle \sum_{\alpha,\beta=1}^3 (\partial v_\alpha/\partial x_\beta)^2 \rangle = \int_0^\infty k^2 E(k) dk$, where $\langle \cdots \rangle$ denotes spatial average. For $E(k) \sim k^{-5/3}$ the integral $\int_0^\infty k^2 E(k) dk$ diverges at the upper limit of integration. Of course, $E(k) \sim k^{-5/3}$ for a large Reynolds number turbulent flow does not apply for
$k \to \infty$. Rather there is a high $k$ cutoff at $k \sim k_c$ due to viscosity, such that for $k > k_c$, $E(k)$ decreases much more rapidly than $k^{-5/3}$ with increasing $k$. This indicates that Lyapunov numbers and vectors are only relevant for finite $\Delta x$ provided that $|\Delta x| \ll k_c^{-1}$, in which case exponential growth of $|\Delta x|$ could indeed be expected. However, for $|\Delta x| \gg k_c^{-1}$, the Lyapunov analysis does not apply, and Richardson’s law is applicable. It can be expected that real atmospheric motions contain a large range of scales including, for example, Kolmogorov like turbulence at small scales. This is also indicated by the use of subgrid scale modeling in weather prediction codes. One thus suspects that the use of a Lyapunov type analysis based on infinitesimal perturbations is problematic for scales modeled in these codes (i.e., scales at the grid scale or above). Thus, if one wants to use a weather prediction code to look at how two weather forecasts with differing initial states diverge with time, a linearized Lyapunov type of analysis would not be expected to be valid, and one should look at the nonlinear solutions evolved from the two initial states. Toth and Kalnay 1993, 1997 introduced a procedure based on finite perturbations of a computational atmospheric state. They call this the method of breeding.

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FIGURE 5. Shown by shades is the local BV-dimension. Contour lines indicate improvement (degradation) in the surface pressure forecast initiated at February 11 0000 UTC. The contour interval is 0.5 hPa. The circle represents the time and geographical regions in which the targeted observations were intended to improve the forecasts.
FIGURE 6. Propagation of the targeted low dimensional region. The geopotential height at 500 hPa pressure level is shown by contour lines. The contour interval is 100 gpm. The circle represents the time and geographical regions in which the targeted observations were intended to improve the forecasts.
FIGURE 7. Shown by shades is the wave packet envelope function for the atmospheric wave packet at the 300 hPa pressure level. The geopotential height at the 300 hPa pressure level is shown by contour lines. The contour interval is 100hPa. The circle represents the time and geographical regions in which the targeted observations were intended to improve the forecasts.
FIGURE 8. Shown by shades is the wave packet envelope function for the signal $s(x)$ at the 300 hPa pressure level. The geopotential height at the 300 hPa pressure level is shown by contour lines. The contour interval is 100hPa. The circle represents the time and geographical regions in which the targeted observations were intended to improve the forecasts.